UNION CURVATURE OF A VECTOR FIELD IN A SUBSPACE*

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ABSTRACT

The differential equations of union curves on a hypersurface V_n immersed in a Riemannian V_{n+1} have been obtained by Springer [1]. These results were generalized later for a subspace in a Riemannian space by Mishra [2]. The author [3] has defined the union curvature of a vector field with respect to a curve on a hypersurface V_n of a Riemannian V_{n+1}.

The purpose of this paper is to consider union curvature of a vector field with respect to a curve in a subspace V_n of a Riemannian V_m.

1. **Subspaces of V_m . Totally indicatrix variety of a vector field.** Consider a subspace V_n of coordinates $x^i (i = 1, \cdots, n)$ and metric

(1.1) $$ds^2 = g_{ij}dx^i dx^j$$

immersed in a Riemannian V_m of coordinates $y^\alpha (\alpha = 1, \cdots, m)$ and metric

(1.2) $$ds^2 = a_{\alpha\beta}dy^\alpha dy^\beta$$

Let $N^\alpha_{\sigma/}$ $(\sigma = n+1, \cdots, m)$ be the contravariant components in the y's of $(m - n)$ independent mutually orthogonal unit vectors normal to V_n, then the following relations hold [4]

(1.3) $$a_{\alpha\beta}N^\alpha_{\mu/}N^\beta_{\mu/} = 1$$

(1.4) $$a_{\alpha\beta}N^\alpha_{\mu/}N^\beta_{\sigma/} = 0 \qquad (\mu \neq \sigma)$$

(1.5) $$a_{\alpha\beta}N^\alpha_{\sigma/}y^\beta_{,i} = 0 \qquad (\sigma = n+1, \cdots, m)$$

(1.6) $$y^\alpha_{;ij} = \sum_\sigma \Omega_{\sigma/ij}N^\alpha_{\sigma/}$$

where the coefficients $\Omega_{\sigma/ij}$ are symmetric covariant tensor of the second order.

Let V be a field of unit vectors in V_n such that if $x^i = x^i(s)$ defines a curve C in V_n, there is associated a unit vector with each point of the curve. If v^α and p^i are the components of V in the y's and the x's respectively, then $v^\alpha = y^\alpha_{,i}p^i$.

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We have the following relation for the derived vector T^α of V in V_m with respect to the curve C [3,4,5]

$$(1.7) \qquad T^\alpha = \sum_\sigma \left(\Omega_{\sigma/ij}p^i\frac{dx^j}{ds}\right)N^\alpha_{\sigma/} + {}_vk_a q^i y^\alpha_{,i}$$

As an analogue to the osculating variety of a curve C we define the totally indicatrix variety of a vector field V with respect to the curve C as that determined by the vector field V and its derived vector in V_m with respect to C. Further let us consider a set of $(m-n)$ congruences of curves one curve of each of which passes through each point of V_n. Let $\lambda^\alpha_{\tau/}$ be the unit vector in the direction of a curve of the congruence which is in general not normal to V_n and therefore may be specified by

$$(1.8) \qquad \lambda^\alpha_{\tau/} = t^i_{\tau/}y^\alpha_{,i} + \sum_\sigma c_{\sigma\tau/}N^\alpha_{\sigma/}$$

where $t^i_{\tau/}$ and $c_{\sigma\tau/}$ are parameters.

It is easy to verify that [2]

$$(1.9) \qquad g_{ij}t^i_{\tau/}t^j_{\tau/} + \sum_\sigma c^2_{\sigma\tau/} = 1$$

$$1.10 \qquad \cos\theta_{\sigma\tau/} = c_{\sigma\tau/}$$

and

$$(1.11) \qquad \cos\alpha_{\tau/} = g_{ij}t^i_{\tau/}p^j$$

where $\theta_{\sigma\tau/}$ and $\alpha_{\tau/}$ are the inclinations of the vector λ_τ to the vectors $N_{\sigma/}$ and V respectively.

2. **Union curvature of a vector field.** If we consider an ndicatrix in V_m with the above direction $\lambda^\alpha_{\tau/}$, then the condition that it is an indicatrix variety of the vector field V requires

$$(2.1) \qquad \lambda^\alpha_{\tau/} = u_{\tau/}y^\alpha_{,i}p^i + w_{\tau/}T^\alpha$$

where $u_{\tau/}$ and $w_{\tau/}$ are parameters to be determined.

From (1.7), (1.8) and (2.1) it follows that

$$t^i_{\tau/}y^\alpha_{,i} + \sum_\sigma c_{\sigma\tau/}N^\alpha_{\sigma/} = u_{\tau/}y^\alpha_{,i}p^i +$$

$$w_{\tau/}\left[\sum_\sigma \left(\Omega_{\sigma/ij}p^i\frac{dn^j}{ds}\right)N^\alpha_{\sigma/} + {}_vk_a q^i y^\alpha_{,i}\right]$$

and therefore we obtain

$$(2.2) \qquad u_{\tau/} = p_h t^h_{\tau/}$$

(2.3)
$$\frac{1}{w_{\tau/}} = \frac{\Omega_{\sigma/ij} p^i \dfrac{dx^j}{ds}}{c_{\sigma\tau/}}$$

and

(2.4)
$$ {}_v k_a q^h + \frac{\Omega_{\sigma/ab} p^a \dfrac{dx^b}{ds}}{c_{\sigma\tau/}} \left(t^i_{\tau/} p_i p^h - t^h_{\tau/} \right) = 0$$

Since (2.3) implies that

$$\frac{1}{w_{\tau/}} = \frac{\Omega_{\sigma/ij} p^i \dfrac{dx^j}{ds}}{c_{\sigma\tau/}} = \frac{\Omega_{\rho/ij} p^i \dfrac{dx^j}{ds}}{c_{\rho\tau/}}$$

therefore (2.4) may also be written as

(2.5)
$$\mu^h_{\tau/} \equiv vk_a q^h + \frac{e({}_v k_n)}{(\sum_\rho c^2_{\rho\tau/})^{1/2}} \sigma^h_{\tau/} = 0$$

where $e({}_v k_n)$, the normal curvature of the vector field V with respect to C in V_n of V_m, and the vector $\sigma^h_{\tau/}$ are defined by

$${}_v k_n^2 = \sum_\sigma \left(\Omega_{\sigma/ab} p^a \frac{dx^b}{ds} \right)^2$$

and

$$\sigma^h_{\tau/} = p_i (t^i_{\tau/} p^h - p^i t^h_{\tau/}).$$

The vector with components $\mu^h_{\tau/}$ will be called the union curvature vector of the vector field V with respect to the curve C relative to the congruence $\lambda_{\tau/}$. Thus union curves of a vector field V relative to a congruence $\lambda_{\tau/}$ are curves along which the union curvature vector $\mu^h_{\tau/}$ is a null vector. Its magnitude which is the union curvature of the vector field V with respect to C relative to the congruence $\lambda_{\tau/}$ is given by

$$\begin{aligned} {}_v k_u^2(\tau) &= \mu^h_{\tau/} \mu_{\tau/h} \\ &= \left({}_v k_a q^h + \frac{e({}_v k_n)}{(\sum_\nu c^2_{\nu\tau/})^{1/2}} \sigma^h_{\tau/} \right) \left({}_v k_a q_h + \frac{e({}_v k_n)}{(\sum_\nu c^2_{\nu\tau/})^{1/2}} \sigma_{\tau/h} \right) \end{aligned}$$

or

(2.6)
$${}_v k_u^2(\tau) = {}_v k_a^{\,2} + \frac{2\,{}_v k_a e({}_v k_n)}{(\sum_\nu c^2_{\nu\tau/})^{1/2}} q^h \sigma_{\tau/h} + \frac{{}_v k^{\,2}}{\sum_\nu c^2_{\nu\tau/}} \sigma^h_{\tau/} \sigma_{\tau/h}.$$

If $e({}_v k_n) = 0$, then (2.6) gives

$${}_v k_u(\tau) = {}_v k_a.$$ Hence:

The union curvature of a vector field V relative to any congruence $\lambda_{\tau|}$ with respect to asymptotic lines of the field V is the associate curvature of the vector field V.

From (2.6) we find that ${}_v k_u(\tau)$ vanishes if ${}_v k_a$ and ${}_v k_n$ are both zero. Thus:

If a curve in V_n is an indicatrix as well as asymptotic line of the vector field V then it is also a union curve of the vector field V relative to any congruence $\lambda_{\tau|}$.

If the congruence be one of normals to V_n in which case $t^i_{\tau|} = 0$, it follows from (2.6) that

$$ {}_v k_u^{(\tau)} = {}_v k_a. $$

Therefore:

A union curve of a vector field V in V_n relative to any normal congruence $\lambda_{\tau|}$ is also an indicatrix of the vector field V.

Making use of (1.10) and (1.11) in (2.6) we obtain for the magnitude ${}_v k_u(\tau)$ of the union curvature of a vector field relative to the congruence $\lambda_{\tau|}$ the following relation in terms of the inclinations $\theta_{v\tau|}$ and $\alpha_{\tau|}$

$$ {}_v k_u(\tau) = {}_v k_a - e({}_v k_n) \left[\frac{\sin^2\alpha_{\tau|} - \sum_v \cos^2\theta_{v\tau|}}{\sum_v \cos^2\theta_{v\tau|}} \right]^{1/2}. \tag{2.7} $$

Taking $m = n + 1$, $N^\alpha_{v|} = N^\alpha$ and the vectors of the field V tangent to the curve C, we find that the formula (2.7) for ${}_v k_u(\tau)$ agrees with the known formula for k_u given by Springer [1]. In this case we obtain

$$ k_u = k_g - k_n \left(\frac{\sin^2\alpha - \cos^2\theta}{\cos^2\theta} \right)^{1/2}. \tag{2.8} $$

If the congruence be one of normals to V_n, then $\cos\theta = 1$, and $\cos\alpha = 0$, therefore (2.8) yields the relation

$$ k_u = k_g $$

which is the known result [1] that corresponding to a normal congruence, the union curves are geodesic curves.

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